Learning and Dynamic Modal Logic

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What do we mean by 'learning'?

General **qualitative** model of (exact) learning:

- on the basis of incoming data consistent with an underlying concept
- learner achieves a **desired type of knowledge** of the underlying concept.

This perspective in various ways generalises many popular learning topics:

- one step updates with an incoming piece of information:
  Belief Revision Theory, Dynamic Epistemic Logic
- particular algorithmic probabilistic methods of automatic improvement:
  Machine Learning, Bayesian Learning, Reinforcement Learning

Gierasimczuk, N., Learning by Erasing in Dynamic Epistemic Logic. LATA 2009.

Gierasimczuk, N., Bridging Learning Theory and Dynamic Epistemic Logic. Synthese 2009.

Gierasimczuk, N., Knowing One's Limits. Logical Analysis of Inductive Inference. PhD thesis, Universiteit van Amsterdam 2010.

Outline

Subset Spaces, Learnability, and Solvability

Topo-characterizations of Learnability and Solvability

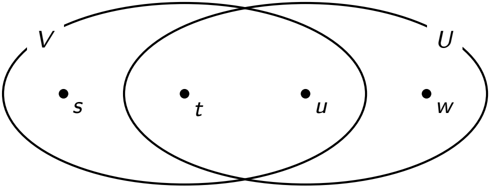
Learning and Modal Logic: There

Learning and Modal Logic: And Back Again

SUBSET SPACE

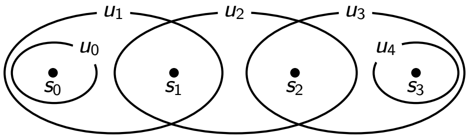
DEFINITION

A **subset space** is $(X, \mathcal{O})$, where $\mathcal{O} \subseteq \mathcal{P}(X)$, X and $\mathcal{O}$ (at most) countable.



Example: Finite Identifiability

Resulting knowledge: Certainty



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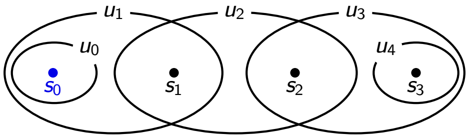

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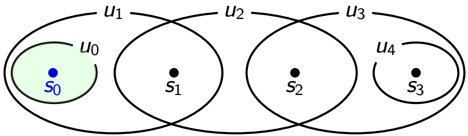

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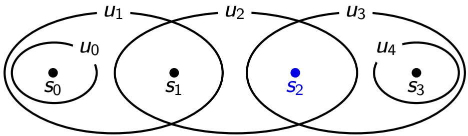

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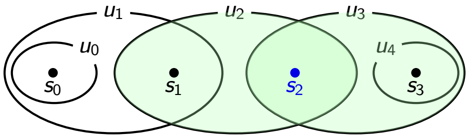
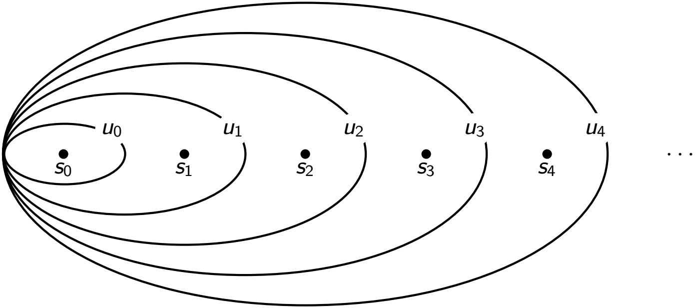
Resulting knowledge: Certainty



Example: Identifiability in the Limit

Resulting knowledge: undefeated belief



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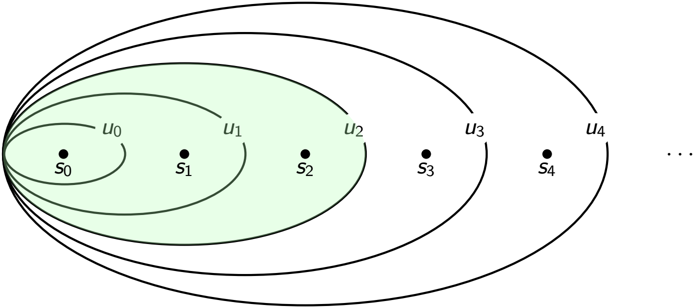
Resulting knowledge: undefeated belief



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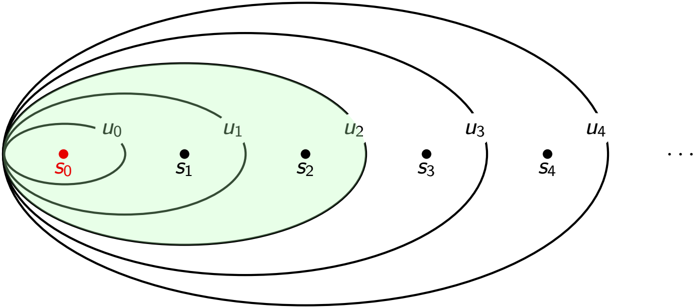
Resulting knowledge: undefeated belief



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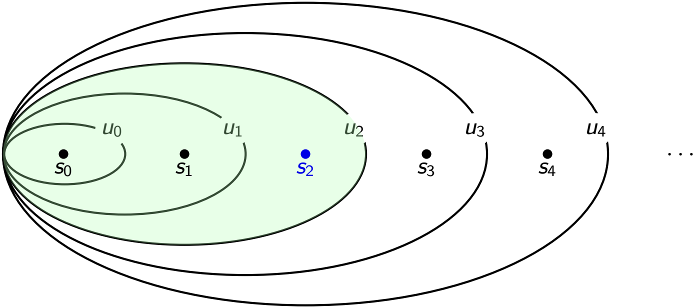
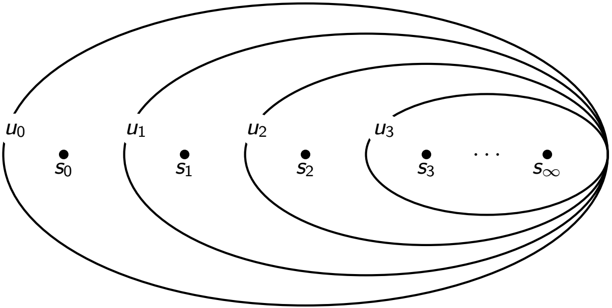
Resulting knowledge: undefeated belief



Example: Non-Identifiability in the Limit



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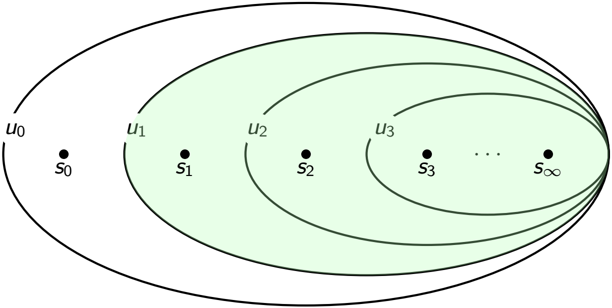

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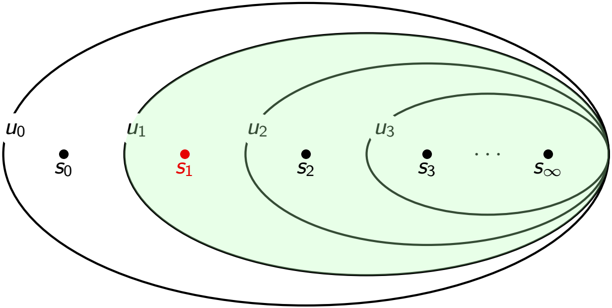

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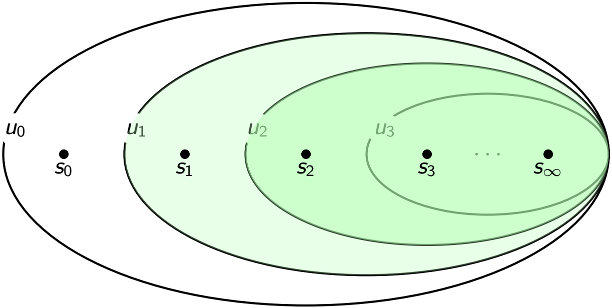

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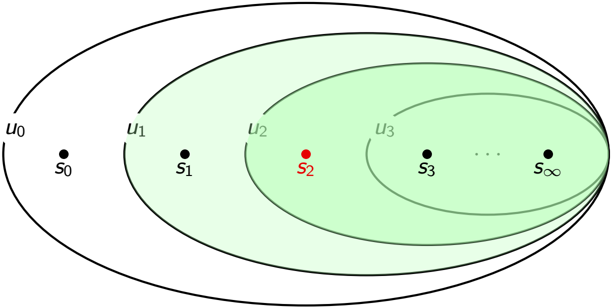


Learning: Streams of Observations

Definition

Let $(X, \mathcal{O})$ be a subset space.

- A **data stream** is an infinite sequence $\vec{O} = (O_0, O_1, \ldots)$ from $\mathcal{O}$.
- A **data sequence** $\vec{O}[n]$ is a finite initial segment of $\vec{O}$ of length $n+1$.

Definition

Take $(X, \mathcal{O})$ and $s \in S$. A data stream $\vec{O}$ is:

- **sound with respect to** s iff every element listed in $\vec{O}$ is true in s.
- **complete with respect to** s iff every observable true in s is listed in $\vec{O}$.

We assume that data streams are sound and complete.

Learning: Learners and Conjectures

Definition

Let $(X, \mathcal{O})$ be a subset space and let σ be a data sequence.

A **learner** L is a function that on σ outputs a conjecture $L(\sigma) \subseteq X$.

Definition

$(X, \mathcal{O})$ is **identified in the limit by** L if for every $x \in X$ and every data stream $\vec{O}$ for x, there is $k \in \mathbb{N}$ s.t.:

$$L(\vec{O}[n]) = \{x\} \text{ for all } n \geq k.$$

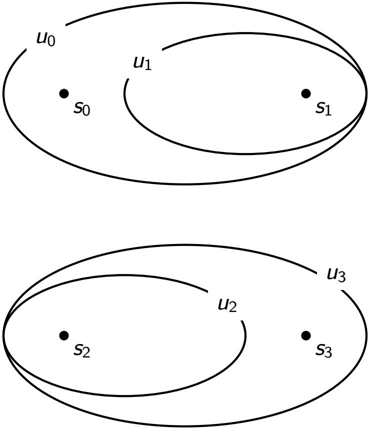
$(X, \mathcal{O})$ is **identifiable in the limit** if it is identified in the limit by a learner L.

Questions, Answers, and Problems

Definition

- A **question** $\mathcal{Q}$ is a partition of X, whose cells A_i are called **answers to** $\mathcal{Q}$.
- Given $x \in A \subseteq X$, $A \in \mathcal{Q}$ is called **the answer to** $\mathcal{Q}$ **at** x, denoted A_x.
- $\mathcal{Q}'$ is a **refinement** of $\mathcal{Q}$ if answers of $\mathcal{Q}$ are disjoint unions of those of $\mathcal{Q}'$.
- A **problem** is a pair $((X, \mathcal{O}), \mathcal{Q})$, where $\mathcal{Q}$ is a question over X.
- $((X, \mathcal{O}), \mathcal{Q}')$ is a **refinement** of $((X, \mathcal{O}), \mathcal{Q})$ if $\mathcal{Q}'$ is a refinement of $\mathcal{Q}$.

Example: Refinements



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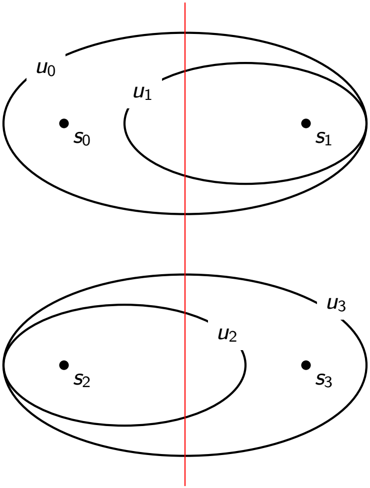

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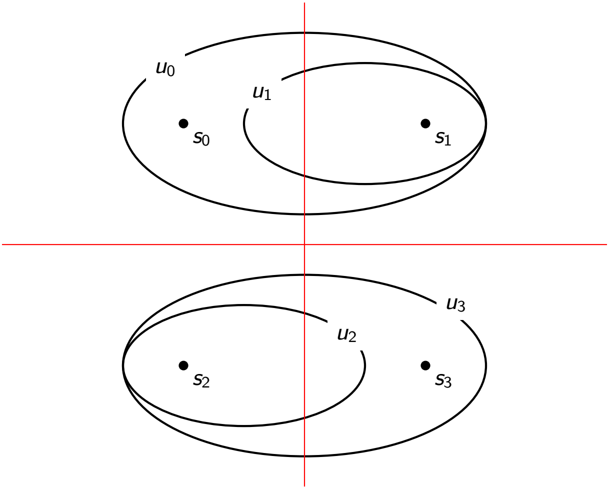

Solving in the Limit

Definition
$((X, \mathcal{O}), \mathcal{Q})$ is **solved in the limit by** L if for every $x \in X$ and every data stream $\vec{O}$ for x, there is $k \in \mathbb{N}$ s.t.:

$$L(\vec{O}[n]) \subseteq A_x \text{ for all } n \geq k.$$

$((X, \mathcal{O}), \mathcal{Q})$ is **solvable in the limit** if solved in the limit by a learner L.

Outline

Subset Spaces, Learnability, and Solvability

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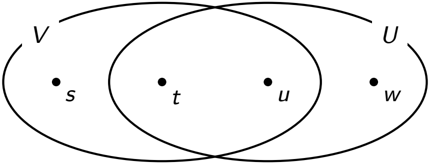
Definition

A a subset space $(X, \mathcal{O})$ is topological if:

1. $\emptyset \in \mathcal{O}$,
2. $X \in \mathcal{O}$,
3. for any $Y \subseteq \mathcal{O}$, $\bigcup Y \in \mathcal{O}$, and
4. for any finite $Y \subseteq \mathcal{O}$, we have $\bigcap Y \in \mathcal{O}$.



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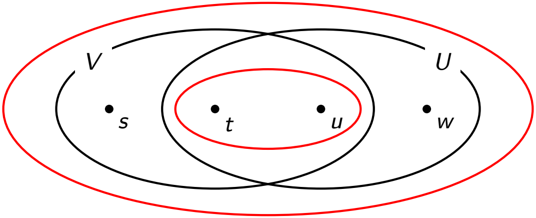
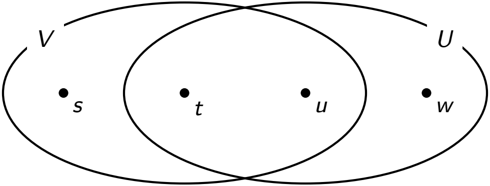
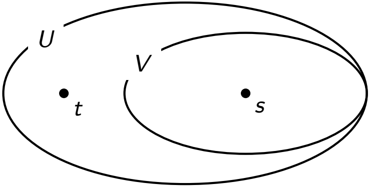


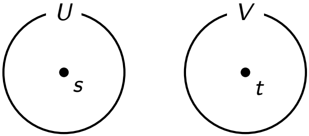
Separability by observations: Illustration



(A) t and u are not separable



(B) weakly separated space, $T0$



(C) strongly separated space, $T1$

Locally Closed and Constructible Sets

Definition

A topological space $(X, \mathcal{O})$ is T_d iff
for every $x \in X$ there is a $U \in \mathcal{O}$ such that $U \setminus \{x\} \in \mathcal{O}$.

T_d is a separation property between $T0$ and $T1$.

Definition

A set A is **locally closed** if $A = U \cap C$, where U is open and C is closed.

Characterization of Solvability in the Limit

Theorem
$((X, \mathcal{O}), \mathcal{Q})$ *is solvable in the limit iff* $\mathcal{Q}$ *has a locally closed refinement.*

Corollary
$(X, \mathcal{O})$ *is identifiable in the limit iff it is* T_d.

A. Baltag, N. Gierasimczuk, S. Smets, On the solvability of inductive problems: a study in epistemic topology, TARK 2015.

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Relational semantics for modal logic

Definition (Syntax)

Let P be a countable set of propositional symbols, $p \in P$.

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi$$

Definition (Semantics)

Given a model $M = (W, R, v)$, where $R \subseteq W \times W$, $v : P \to \wp(W)$, $x \in W$:

$$\begin{array}{lll}
M, x \models p & \text{iff} & x \in v(p) \text{ for each } p \in P \\
M, x \models \neg\varphi & \text{iff} & \text{not } M, x \models \varphi \\
M, x \models \varphi \wedge \psi & \text{iff} & M, x \models \varphi \text{ and } M, x \models \psi \\
M, x \models \Box\varphi & \text{iff} & \text{for all } y \in W\text{: if } xRy \text{ then } M, y \models \varphi
\end{array}$$

Some Axioms and Their Epistemic Interpretation

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box\varphi$

Axioms

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ (omniscience)

(T) $\Box\varphi \rightarrow \varphi$ (truthfullness/reflexivity)

(D) $\Box\varphi \rightarrow \neg\Box\neg\varphi$ (consistency/seriality)

(4) $\Box\varphi \rightarrow \Box\Box\varphi$ (positive introspection/transitivity)

(5) $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$ (negative introspection/Euclidean-ness)

Some Axioms and Their Epistemic Interpretation

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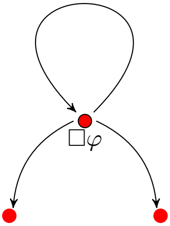
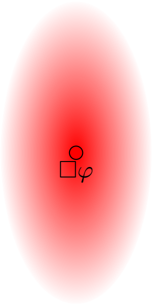
Ax is a logic of a class of models $\mathcal{M}$ iff *Ax* is sound and complete wrt $\mathcal{M}$.

Topological Interpretations

Relational $\Box$ vs Topological $\Box := \mathit{Int}$





Topological Topo-semantics for Modal Logic

Definition (Syntax)

Let P be a countable set of propositional symbols, $p \in P$.

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi$$

Definition

A **topological model** (or a topo-model) $M = (X, \mathcal{O}, v)$ is a topological space $(X, \mathcal{O})$ together with a valuation function $v : P \to \mathcal{P}(X)$.

Definition (Semantics)

Given a topological model $M = (X, \mathcal{O}, v)$ and a state $x \in X$:

$M, x \models p$	iff	$x \in v(p)$ for each $p \in P$
$M, x \models \neg\varphi$	iff	not $M, x \models \varphi$
$M, x \models \varphi \wedge \psi$	iff	$M, x \models \varphi$ and $M, x \models \psi$
$M, x \models \Box\varphi$	iff	there is $U \in \tau(x \in U$ and for all $y \in U$: $M, y \models \varphi)$

Sound and Complete Topo-Axiomatizations

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box\varphi$

Axioms

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

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(4) $\Box\varphi \to \Box\Box\varphi$

S4 is the topo-logic of all topological spaces (McKinsey & Tarski 1944).

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$

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Axioms

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(T) $\Box\varphi \to \varphi$

S4=Topo

(4) $\Box\varphi \to \Box\Box\varphi$

S4 is the topo-logic of all topological spaces (McKinsey & Tarski 1944).

What about T_d-spaces (identifiable in the limit)?

T_d is not topo-definable.

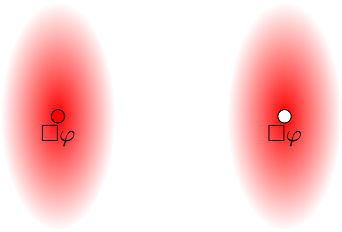
The identifiability-adequate notion of belief is not *topo*-definable.

What about T_d-spaces (identifiable in the limit)?

T_d is not topo-definable.

The identifiability-adequate notion of belief is not *topo*-definable.

But let us, on a whim, change the way we view $\Box$.



DEFINITION (SEMANTICS)

Given a topological model $M = (X, \mathcal{O}, v)$ and a state $x \in X$:

$$
\begin{array}{lll}
M, x \models_d p & \text{iff} & x \in v(p) \\
M, x \models_d \neg\varphi & \text{iff} & \text{not } M, x \models_d \varphi \\
M, x \models_d \varphi \wedge \psi & \text{iff} & M, x \models_d \varphi \text{ and } M, x \models_d \psi \\
M, x \models_d \Box\varphi & \text{iff} & \exists U \in \tau (x \in U \ \& \ \forall y \in U - \{x\} \ M, y \models_d \varphi)
\end{array}
$$

Sound and Complete d-Axiomatizations

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box\varphi$

Axioms

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

(T) $\Box\varphi \to \varphi$

(D) $\Box\varphi \to \neg\Box\neg\varphi$

(4) $\Box\varphi \to \Box\Box\varphi$

(5) $\neg\Box\varphi \to \Box\neg\Box\varphi$

(w) $(\varphi \wedge \Box\varphi) \to \Box\Box\varphi$

(GL) $\Box(\Box\varphi \to \varphi) \to \Box\varphi$

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box\varphi$

Axioms

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

(D) $\Box\varphi \to \neg\Box\neg\varphi$

(5) $\neg\Box\varphi \to \Box\neg\Box\varphi$

(w) $(\varphi \wedge \Box\varphi) \to \Box\Box\varphi$

wKD45=dense

wKD45 is the d-logic of dense spaces.

Sound and Complete d-Axiomatizations

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box\varphi$

Axioms

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

(D) $\Box\varphi \to \neg\Box\neg\varphi$

(4) $\Box\varphi \to \Box\Box\varphi$

(5) $\neg\Box\varphi \to \Box\neg\Box\varphi$

KD45=DSO

KD45 is the d-logic of DSO-spaces.

Sound and Complete d-Axiomatizations

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box\varphi$

Axioms

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

GL=scattered

(GL) $\Box(\Box\varphi \to \varphi) \to \Box\varphi$

GL is the d-logic of scattered spaces.

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box\varphi$

Axioms

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

wK4=Topo

(w) $(\varphi \wedge \Box\varphi) \to \Box\Box\varphi$

wK4 is the d-logic of all topological spaces.

Sound and Complete d-Axiomatizations

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box\varphi$

Axioms

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

$\text{K4} = T_d$

(4) $\Box\varphi \to \Box\Box\varphi$

Finally, K4 is the d-logic of all T_d-spaces!

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box\varphi$

Axioms

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

And so what...?

(4) $\Box\varphi \to \Box\Box\varphi$

Finally, K4 is the d-logic of all T_d-spaces!

Another way

Get dynamic!

Baltag, A., Gierasimczuk, N., Özgün, A., Vargas Sandoval, A.L., and Smets S., A dynamic logic for learning theory. J. Log. Algebr. Meth. Program. 2019.

Starting point: Subset Space Logic

Definition (Syntax)

Let P be a countable set of propositional symbols and $p \in P$.

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \Box\varphi$$

Definition

An **intersection model** $M = (X, \mathcal{O}, v)$ is an intersection space $(X, \mathcal{O})$ together with a valuation function $v : P \to \mathcal{P}(X)$.

Definition (Semantics)

Given an intersection model $M = (X, \mathcal{O}, v)$, $U \in \mathcal{O}$, and $x \in U$:

$$\begin{array}{lll}
M, x, U \models p & \text{iff} & x \in v(p) \\
M, x, U \models \neg\varphi & \text{iff} & M, x, U \not\models \varphi \\
M, x, U \models \varphi \wedge \psi & \text{iff} & M, x, U \models \varphi \text{ and } M, x, U \models \psi \\
M, x, U \models K\varphi & \text{iff} & \forall y \in U \;\; M, y, U \models \varphi \\
M, x, U \models \Box\varphi & \text{iff} & \forall O \in \mathcal{O} \;\; \text{if } x \in O \subseteq U \text{ then } M, x, O \models \varphi
\end{array}$$

A. Dabrowski, L.S. Moss, R. Parikh, Topological reasoning and the logic of knowledge, Annals of Pure and Applied Logic 1996.

A Dynamic Logic for Learning Theory (DLLT)

Definition (Syntax)

Let p and o be drawn from countable sets of propositional and observational symbols, P and O respectively.

$$\varphi := p \mid o \mid L(\vec{o}) \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \Box\varphi \mid [o]\varphi$$

(DLLT): Learning Models

Definition

A *learning model* $\mathcal{M} = (X, \mathcal{O}, \mathbb{L}, v)$ consists of:

- an intersection space $(X, \mathcal{O})$, as before.
- a learner $\mathbb{L} : \mathcal{O} \to \mathcal{P}(X)$, s.t.:
  1. $\mathbb{L}(O) \subseteq O$, and
  2. if $O \neq \emptyset$ then $\mathbb{L}(O) \neq \emptyset$.

     (Additionally: $\mathbb{L}(\vec{O}) := \mathbb{L}(\bigcap \vec{O})$, where $\bigcap \vec{O} := O_1 \cap \ldots \cap O_n$).
- a valuation map $v : P \cup O \to \mathcal{P}(X)$

(DLLT): Semantics

Definition (Semantics)

Given a learning model $\mathcal{M} = (X, \mathcal{O}, \mathbb{L}, v)$, $U \in \mathcal{O}$, and $x \in U$:

$$
\begin{array}{lll}
M, x, U \models p & \text{iff} & x \in v(p) \\
M, x, U \models o & \text{iff} & x \in v(o) \\
M, x, U \models L(o_1, \ldots, o_n) & \text{iff} & x \in \mathbb{L}(U, v(o_1), \ldots, v(o_n)) \\
M, x, U \models \neg\varphi & \text{iff} & M, x, U \not\models \varphi \\
M, x, U \models \varphi \wedge \psi & \text{iff} & M, x, U \models \varphi \text{ and } M, x, U \models \psi \\
M, x, U \models K\varphi & \text{iff} & \forall y \in U \; M, y, U \models \varphi \\
M, x, U \models \Box\varphi & \text{iff} & \forall O \in \mathcal{O} \;\; \text{if } x \in O \subseteq U \text{ then } M, x, O \models \varphi \\
M, x, U \models [o]\varphi & \text{iff} & x \in v(o) \text{ implies } M, x, U \cap v(o) \models \varphi
\end{array}
$$

Abbreviations

- $\bigwedge \vec{o} \ := \ o_1 \wedge \ldots \wedge o_n \ (\bigwedge \lambda := \top)$
- $\vec{o} \Leftrightarrow \vec{u} \ := \ K\,((\bigwedge \vec{o}) \leftrightarrow (\bigwedge \vec{u}))$
- $[\vec{o}]\varphi \ := \ [o_1]\ldots[o_n]\varphi \ \ ([\lambda]\varphi := \varphi)$; similarly for $\langle \vec{o} \rangle$)
- $B^{\vec{o}}\varphi \ := \ K(L(\vec{o}) \to \varphi)$
- $B\varphi \ := \ B^{\lambda}\varphi$

	Basic axioms:
(P)	all instantiations of propositional tautologies
(K_K)	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
(T_K)	$K\varphi \to \varphi$
(4_K)	$K\varphi \to KK\varphi$
(5_K)	$\neg K\varphi \to K\neg K\varphi$
($K_{[o]}$)	$[o](\psi \to \chi) \to ([o]\psi \to [o]\chi)$
	Basic rules:
(MP)	From φ and $\varphi \to \psi$, infer ψ
(Nec_K)	From φ, infer $K\varphi$
($Nec_{[o]}$)	From φ, infer $[o]\varphi$

Learning axioms:

(CC)	$(\bigwedge \vec{o}) \rightarrow \langle K \rangle L(\vec{o})$	Consistency
(EC)	$(\vec{o} \Leftrightarrow \vec{u}) \rightarrow (L(\vec{o}) \leftrightarrow L(\vec{u}))$	Extensionality
(SP)	$L(\vec{o}) \rightarrow \bigwedge \vec{o}$	Success Postulate

Reduction axioms:

(R_p)	$[o]p \leftrightarrow (o \rightarrow p)$
(R_u)	$[o]u \leftrightarrow (o \rightarrow u)$
(R_L)	$[o]L(\vec{u}) \leftrightarrow (o \rightarrow L(o, \vec{u}))$
($R_\neg$)	$[o]\neg\psi \leftrightarrow (o \rightarrow \neg[o]\psi)$
(R_K)	$[o]K\psi \leftrightarrow (o \rightarrow K[o]\psi)$
($R_\Box$)	$[o]\Box\psi \leftrightarrow \Box[o]\psi$

Effort axiom and rule:

($\Box$-Ax)	$\Box\varphi \rightarrow [\vec{o}]\varphi$, for $\vec{o} \in O^*$
($\Box$-Rule)	From $\psi \rightarrow [o]\varphi$, infer $\psi \rightarrow \Box\varphi$, where $o \notin O_\psi \cup O_\varphi$

Completeness

Theorem

DLLT is sound and complete with respect to the class of learning models.

Expressivity (of learning concepts)

Proposition

- $\mathcal{M}, x, U \models \Diamond Kp$ *iff p is* **learnable with certainty** *at x.*

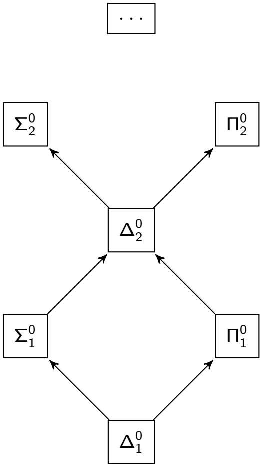
- $\mathcal{M} \models p \to \Diamond Kp$ *iff p is* **verifiable with certainty**.
- $\mathcal{M} \models \neg p \to \Diamond K\neg p$ *iff p is* **falsifiable with certainty**.

- $\mathcal{M}, x, U \models \Box Bp$ *iff* $\mathbb{L}$ *has* **undefeated belief** *in p at x.*
- $\mathcal{M}, x, U \models p \wedge \Box Bp$ *iff* $\mathbb{L}$ *has* **inductive knowledge** *of p at x.*
- $\mathcal{M}, x, U \models p \wedge \Diamond\Box Bp$ *iff p is* **inductively learnable** *by* $\mathbb{L}$ *at x.*

- $\mathcal{M} \models p \to \Diamond\Box Bp$ *iff p is* **verifiable in the limit** *by* $\mathbb{L}$.
- $\mathcal{M} \models \neg p \to \Diamond\Box B\neg p$ *iff p is* **falsifiable in the limit** *by* $\mathbb{L}$.

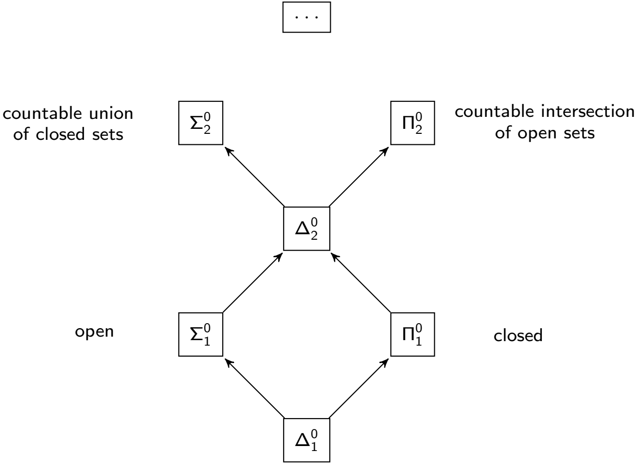
Descriptive Set Theory

Borel Hierarchy



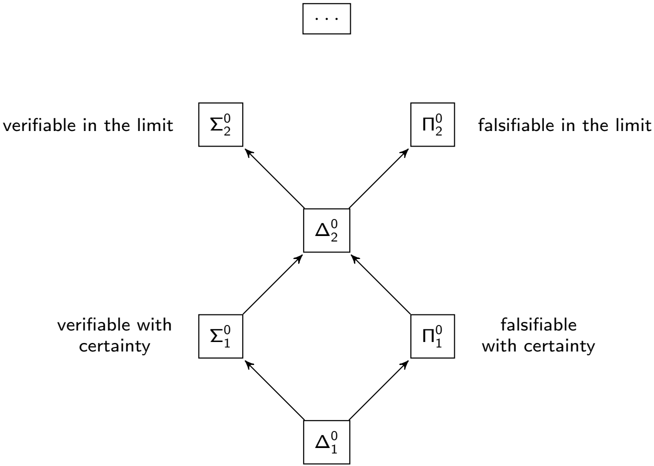
Descriptive Set Theory

Borel Hierarchy



Descriptive Set Theory

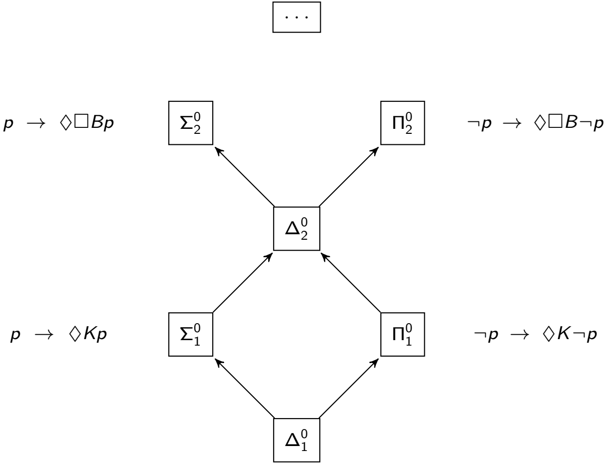
Borel Hierarchy



K.T. Kelly, The Logic of Reliable Inquiry, Oxford University Press, 1996.

Descriptive Set Theory

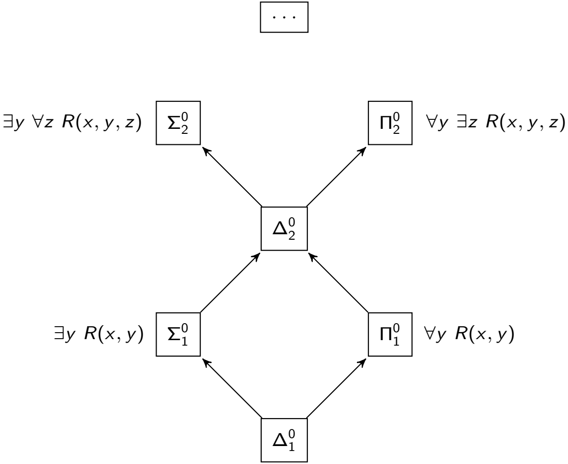
DLLT-definability

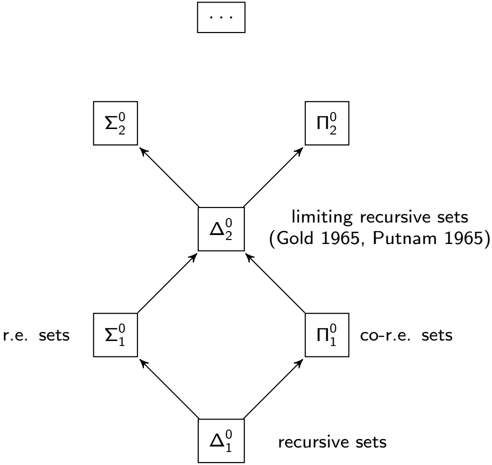


Outline

Effective Descriptive Set Theory

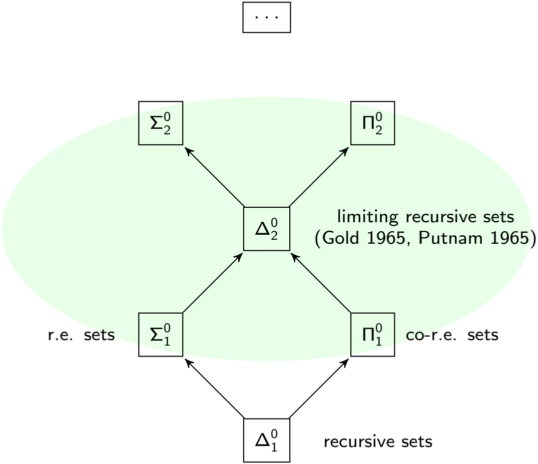
Kleene-Mostowski Hierarchy




. . .
Σ^0_2
Π^0_2
Δ^0_2
limiting recursive sets
(Gold 1965, Putnam 1965)
r.e. sets
Σ^0_1
Π^0_1
co-r.e. sets
Δ^0_1
recursive sets

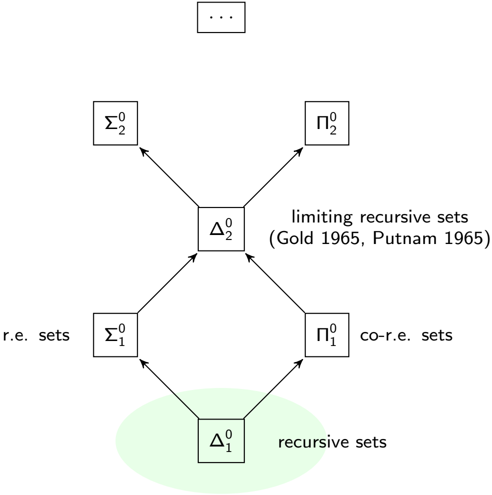
Effective Descriptive Set Theory

Kleene-Mostowski Hierarchy



Effective Descriptive Set Theory

Kleene-Mostowski Hierarchy



Indexed families of recursive sets

Definition

An **indexed family of recursive sets** is a class $\mathcal{C} = (S_i)_{i \in \mathbb{N}}$ for which a computable function $f : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ exists that uniformly decides $\mathcal{C}$, i.e.,

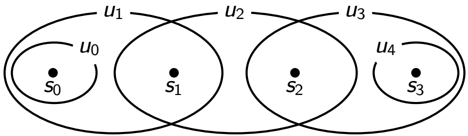
$$f(i, w) = \begin{cases} 1 & \text{if } w \in S_i, \\ 0 & \text{if } w \notin S_i. \end{cases}$$

.

Angluin, D., Inductive inference of formal languages from positive data. Information and Control 1980.

Motivation: Finite Identifiability

Resulting knowledge: Certainty



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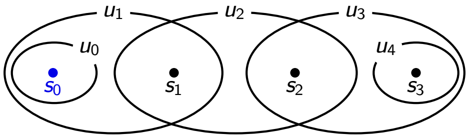
Resulting knowledge: Certainty



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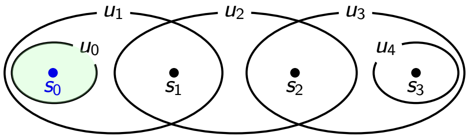
Resulting knowledge: Certainty



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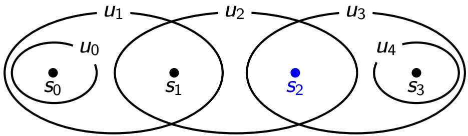

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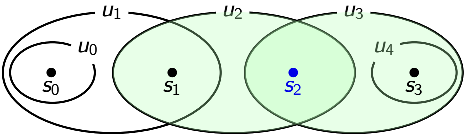

Texts and Learners

Definition

A **text** τ for a set S is an infinite sequence of all and only the elements from S. τ_n is the n-th element of τ and $\tau \upharpoonright n$ is the sequence $(\tau_0, \tau_1, \ldots, \tau_{n-1})$;

Definition

A **learning function**, L, is a recursive map from finite data sequences to indices of languages, $L : \mathbb{N}^* \to \mathbb{N} \cup \{\uparrow\}$.

Intuitively: If $L(\tau \upharpoonright n) = i$ then L conjectures that the language is S_i.
L can refuse to give a natural number answer, in that case the output is $\uparrow$;

Texts and Learners

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Intuitively: If $L(\tau \upharpoonright n) = i$ then L conjectures that the language is S_i.
L can refuse to give a natural number answer, in that case the output is $\uparrow$;

Definition

A learning function L is **(at most) once defined** on $\mathcal{C}=(S_i)_{i\in\mathbb{N}}$ iff for any text τ for a language in $\mathcal{C}$ and $n, k \in \mathbb{N}$ such that $n{\neq}k$: $L(\tau \upharpoonright n){=}\uparrow$ or $L(\tau \upharpoonright k){=}\uparrow$.

Finite identifiability

Definition

Let $\mathcal{C} = (S_i)_{i \in \mathbb{N}}$ as before. A learning function L:

1. finitely identifies S_i in $\mathcal{C}$ on τ iff

   L is once defined on τ and the defined value is j, such that $S_i = S_j$;
2. finitely identifies S_i in $\mathcal{C}$ iff it finitely identifies S_i on every τ for S_i;
3. finitely identifies $\mathcal{C}$ iff it finitely identifies every S_i in $\mathcal{C}$.

$\mathcal{C}$ is **finitely identifiable** iff there is an L that finitely identifies $\mathcal{C}$.

Learning powers

$$\mathsf{FIN} \subset \mathsf{SMON} \subset \mathsf{MON} \subset \mathsf{WMON} \subset \mathsf{LIM}$$

Learning powers

FIN $\subset$ SMON $\subset$ MON $\subset$ WMON $\subset$ LIM

- Strong-monotonic learning (SMON):
  $S_{L(\tau\upharpoonright n)} \subseteq S_{L(\tau\upharpoonright(n+k))}$
- Monotonic learning (MON):
  $S_{L(\tau\upharpoonright n)} \cap S_i \subseteq S_{L(\tau\upharpoonright(n+k))} \cap S_i$
- Weak-monotonic learning (WMON), i.e., conservative learning:
  if $\text{set}(\tau\upharpoonright(n+k)) \subseteq S_{L(\tau\upharpoonright n)}$, then $S_{L(\tau\upharpoonright n)} \subseteq S_{L(\tau\upharpoonright(n+k))}$

Lange, S. and Zeugmann, T., Types of Monotonic Language Learning and their Characterization. COLT 1992.

Zeugmann, T., Lange, S. and Kapur, S., Characterizations of monotonic and dual monotonic language learning. Information and Computation 1995.

Learning powers

$$\text{FIN} \subset \text{SMON} \subset \text{MON} \subset \text{WMON} \subset \text{LIM}$$

Learning powers

FAST $\subset$ FIN $\subset$ SMON $\subset$ MON $\subset$ WMON $\subset$ LIM

Characterization of finite identifiability

Definition

A set D_i is a **definite finite tell-tale set** (DFTT) for $S_i \in \mathcal{C}$ if

1. $D_i \subseteq S_i$,
2. D_i is finite, and
3. for any index $j \neq i$, if $D_i \subseteq S_j$ then $S_i = S_j$.

Theorem

A family $\mathcal{C}=(S_i)_{i\in\mathbb{N}}$ is finitely identifiable iff there is an effective procedure $\mathcal{D} : \mathbb{N} \to \mathcal{P}^{<\omega}(\mathbb{N})$, given by $n \mapsto \mathcal{D}_n$, that on input i produces a definite finite tell-tale of S_i.

Mukouchi,Y., Characterization of Finite Identification. Analogical and Inductive Inference 1992.

Lange, S. and Zeugmann, T., Types of Monotonic Language Learning and their Characterization, COLT 1992.

Fastest Learning

The fastest learner finitely identifies a language S_i as soon as any DFTT for it has been enumerated.

Definition
Let $\mathbb{D}_i$ be a set of all DFTTs of $S_i \in \mathcal{C}$.
Let $\mathcal{C}$ be an indexed family of recursive sets. $\mathcal{C}$ is **finitely identifiable in the fastest way** if and only if there is a learning function L s.t.:

$$L(\tau \restriction n) = i \quad \text{iff} \quad \exists D_i^j \in \mathbb{D}_i \; D_i^j \subseteq \text{set}(\tau \restriction n) \;\& \\ \neg \exists D_i^k \in \mathbb{D}_i \; D_i^k \subseteq \text{set}(\tau \restriction n-1).$$

We will call such L a **fastest learning function**.

Finite identifiability and fastest learning

Not every finitely identifiable class is identified by a fastest learner.

Theorem

$\mathcal{C}$ exists that is finitely identifiable, but not in the fastest way.

N. Gierasimczuk, D. de Jongh, On the complexity of conclusive update, The Computer Journal 2013.

Conclusions

- Dynamic Modal Logic treatment of learnability.
- **Topological perspective on knowledge is a bridge between modal logic, learning theory, and computability.**
- A new, more restrictive kind of finite identification.
- **Even if computable convergence to certainty is possible, it may not be computably reached at the first instant in which objective ambiguity disappears.**

The End

Thank you!